ON LIMIT CYCLES APPEARING BY POLYNOMIAL PERTURBATION OF DARBOUXIAN INTEGRABLE SYSTEMS

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ABSTRACT. We prove an existential finiteness result for integrals of rational 1-forms over the level curves of Darbouxian integrals.

1. LIMIT CYCLES BORN BY PERTURBATION OF INTEGRABLE SYSTEMS

1.1. **Poincaré–Pontryagin integral.** Limit cycles (isolated periodic trajectories) of polynomial planar vector fields can be produced by perturbing integrable systems which have nested continuous families of non-isolated periodic trajectories. The number and position of limit cycles born in such perturbations is determined by the number and position of *isolated zeros* of the Poincaré–Pontryagin integral of the dissipation form along the closed periodic orbits of the non-perturbed integrable vector field. The *infinitesimal Hilbert problem* is to place an upper bound for the number of isolated zeros of this integral. This problem was repeatedly formulated as a relaxed form of the Hilbert 16th problem by V. Arnold [1].

Instead of polynomial vector fields, it is more convenient to deal with (singular) foliations of the real plane $\mathbb{R}^2$ by solutions of *rational* Pfaffian equations $\theta = 0$, where θ is a 1-form on $\mathbb{R}^2$ with rational coefficients (since only the distribution of null spaces of the form θ makes geometric sense, one can always replace the rational form by a polynomial one). The foliation $\mathcal{F}$ is (Darbouxian) *integrable*, if the form is *closed*, $d\theta = 0$. An integrable foliation always has a "multivalued" first integral of the form $f(x, y) = \exp r(x, y) \cdot \prod_j p_j(x, y)^{\lambda_j}$, where $r(x, y)$ is a rational function and p_j polynomials in x, y, which are involved in the in general non-rational powers λ_j. A particular case of the integrable foliations consists of *Hamiltonian foliations* defined by the *exact polynomial* 1-form $\theta = dh$, $h \in \mathbb{R}[x, y]$.

If $L \subset \mathbb{R}^2$ is a compact smooth leaf (oval) of an integrable foliation $\theta = 0$ belonging to the level curve $\{f = a\} \subset \mathbb{R}^2$, then all nearby leaves of this foliation are also closed and form a continuous family of ovals belonging to the level curves $\{f = t\}$, $t \in (\mathbb{R}^1, a)$. The ovals from this family, denoted by L_t, are uniquely parameterized by the values of the real variable t varying

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in an open interval $\alpha < t < \beta$. Their union is an annulus $A_{\alpha,\beta}$ bounded by the Hausdorff limits $\overline{L}_\alpha = \lim_{t\to\alpha+0} L_t$ and $\overline{L}_\beta = \lim_{t\to\beta-0}$. The limits themselves may be not leaves of the foliation, but rather *separatrix polygons* (unions of several leaves and one or more singular points of the foliation $\mathcal{F}$). The polar locus of the form θ is usually a separatrix polygon.

Consider an one-parameter family of planar real foliations $\mathcal{F}_\varepsilon$ defined by the Pfaffian equations

$$\theta + \varepsilon\omega = 0, \qquad \operatorname{Poles}(\omega) \subseteq \operatorname{Poles}(\theta), \tag{1.1}$$

with a closed rational 1-form θ and arbitrary rational 1-form ω. The assumption on the poles guarantees that the perturbation (1.1) does not create additional singularities in the annulus $A_{\alpha,\beta}$ (such ω will be called *admissible* for θ). The *Poincaré–Pontryagin integral*[1] associated with this family, is the function

$$t \mapsto I(t) = \oint_{L_t} \omega, \qquad L_t \subseteq \{f = t\} \text{ an oval, } t \in (\alpha, \beta). \tag{1.2}$$

The following well-known *Poincaré–Pontryagin criterion* establishes connections between the integral (1.2) and limit cycles of the system (1.1): *If the function $I(t)$ has an isolated zero of multiplicity μ at an interior point $a \in (\alpha, \beta)$, then the perturbed foliation $\mathcal{F}_\varepsilon$ for all sufficiently small values of ε and $\delta > 0$ has no more than μ limit cycles in the thin annulus between the ovals $L_{a-\delta}$ and $L_{a+\delta}$.* The assertion is reasonably sharp: under certain additional assumptions of nondegeneracy one can produce exactly μ limit cycles.

Thus bounds for the number of isolated zeros for the integral (1.2) translate into bounds for the number of limit cycles for near-integrable systems.[2] The following conjecture seems to be believed by most experts in the area.

Conjecture 1. *For any pair of rational 1-forms (θ, ω) on the real plane $\mathbb{R}^2$ of degrees n, m respectively, such that $d\theta = 0$ and $\operatorname{Poles}(\omega) \subseteq \operatorname{Poles}(\theta)$, the number of isolated real zeros of the integral (1.2) is bounded by a constant $N = N(n, m)$ depending only on n and m.*

Demonstration (or refutation) of this Conjecture constitutes the *infinitesimal Hilbert Sixteenth problem* [1, 5]. A separate question concerns computability (and practical computation) of the bound $N(n, m)$.

1.2. **Analytic properties of the Poincaré–Pontryagin integral: Hamiltonian versus Darbouxian cases.** The function $I(t)$ defined by the integral (1.2) is obviously real analytic inside the real interval (α, β), and hence the number of isolated zeros is necessarily finite on any compact subinterval

[1]The function $I(t)$ is sometimes called Melnikov function or (misleadingly) the Abelian integral, see below.

[2]However, the above Poincaré–Pontryagin criterion does not apply to limit cycles born from separatrix polygons $\overline{L}_\alpha, \overline{L}_\beta$ which bound the annuli of periodic orbits of the integrable systems. This bifurcation requires much deeper analysis that will not be discussed here.

of this interval. Yet this circumstance neither implies any explicit upper bound, nor rules out the accumulation of infinitely many isolated roots to the boundary points α, β of the interval.

Most studies of the function $I(t)$ are based on its analytic continuation from the real interval (α, β) to the complex domain. Already at this step drastic differences between Hamiltonian and general integrable cases occur.

1.2.1. *Abelian integrals.* If the initial foliation $\mathcal{F} = \{\theta = 0\}$ is Hamiltonian, i.e., if $\theta = dh$, $h \in \mathbb{R}[x, y]$, then ω must also be polynomial and the function $I(t)$ is an *Abelian integral*, integral of a polynomial 1-form over a cycle on *algebraic curve*. Such integrals possess the following properties.

(i) The integral $I(t)$ admits analytic continuation on the entire Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ as a multivalued function ramified over finitely many points. For a generic Hamiltonian h these points are exactly the (complex) critical values of h.

(ii) $I(t)$ is (a linear combination of coordinates of) a solution of a Fuchsian linear system on $\mathbb{C}P^1$.

(iii) Near each singular point t_0 of the Fuchsian system, $I(t)$ admits a local representation as the finite sum $I(t) = \sum_{j,k} a_{jk}(t)\,(t - t_0)^{\lambda_j} \ln^k(t - t_0)$ with the coefficients a_{jk} holomorphic at the point t_0 and finitely many rational exponents $\lambda_j \in \mathbb{Q}$.

The last property alone suffices to guarantee that Abelian integrals are *non-oscillating*: any such integral has only finitely many real isolated zeros on any interval. This can be proved by application of the "derivation-division algorithm" [15]. Moreover, the Fewnomial theory [3] which can be regarded as a multidimensional generalization of the above algorithm, implies the following uniform version of the non-oscillation property, achieved in [4, 17].

Theorem 1 (A. Khovanskii–A. Varchenko, 1984)**.** *The total number of real isolated zeros of Abelian integrals over the level curves of a polynomial h is uniformly bounded over all polynomial 1-forms ω and all Hamiltonians h sufficiently close to any given combination (h_0, ω_0).*

By the standard compactness and homogeneity arguments, Theorem 1 implies that for any combination of $n, m \in \mathbb{N}$ the number of real isolated zeros of Abelian integrals of forms of degree $\leqslant m$ over Hamiltonians of degree $\leqslant n$ is bounded by a constant $N_0(n, m)$ depending only on n, m. However, this constant is absolutely existential (non-constructive). Other, completely different tools allow to place *explicit* upper bounds on the number of zeros of Abelian integrals which are at least δ-distant from the critical values of the Hamiltonian [9, 10] (the bounds depend on $\delta > 0$).

One can expect similar behavior of the Poincaré–Pontryagin integrals in the case when θ is exact albeit non-polynomial anymore, $\theta = df$, with $f(x, y) = \frac{P(x,y)}{Q(x,y)}$ a rational function, though the subject was not explored systematically.

1.2.2. *Pseudoabelian integrals.* Completely different is the picture if the (unperturbed) closed form θ is *non-exact.* In this article we consider only the simplest case when θ has at most first order poles after extension on $\mathbb{C}P^2$ and hence a Darbouxian integral of the form

$$f(x,y) = \prod_j p_j(x,y)^{\lambda_j}, \qquad \lambda_j \in \mathbb{R}_+,\ j = 1,\dots,m, \tag{1.3}$$

ramified over the separatrix polygon $S(\theta) = \bigcup_j \{p_j = 0\} \subset \mathbb{R}^2$ corresponding to the "critical" value $f = 0$. For lack of better name, we will refer to integrals of the form

$$I(t) = \oint_{L_t} \omega, \qquad L_t \subseteq \{f = t\} \subseteq \mathbb{R}^2, \tag{1.4}$$

where ω is a rational form without singularities on L_t, as *pseudoabelian integrals.*

This particular case already contains all difficulties which distinguish the general integrable case from the Hamiltonian one. The closed level curves $\{f = t\}$ cease to be algebraic unless all ratios λ_j/λ_k are rational, moreover, their complexifications are generically dense in $\mathbb{C}^2$ [6]. The topological methods of continuation of the corresponding integral to the complex domain, which work so nicely in the Hamiltonian case, fail [12], thus the analytical continuation of the corresponding integral $I(t)$ should be achieved by completely different methods. Besides, $I(t)$ is not known to satisfy any reasonable linear or nonlinear differential equation of finite order, which makes impossible applications of the methods from [9, 10]. For example, the local representation (1.5) already implies that $I(t)$ is not a solution of a Fuchsian ordinary differential equation of finite order.

Even the most basic among the properties of the Abelian integrals, the local analytic representation, fails for pseudoabelian integrals. More precisely, it can be shown that if the separatrix curve $S(\theta)$ has only normal crossings and the collection of Darboux exponents $[\lambda_1 : \cdots : \lambda_n]$ satisfies certain generic *arithmetic* properties, then the pseudoabelian integral $I(t)$ can be locally near $t = 0$ represented as a composition,

$$I(t) = F(t, t^{\mu_1}, \cdots, t^{\mu_n}), \qquad \mu_1, \dots, \mu_n \in \mathbb{R}_+, \tag{1.5}$$

with a real analytic function F of $n + 1$ variables. This observation, due to C. Moura [8], suffices to prove non-oscillatory behavior of pseudoabelian integrals, but under the above arithmetical restrictions on the Darboux exponents. Such form of arithmetics-conditioned finiteness is not unknown in the general theory of o-minimal structures, see [16]. Clearly, if this arithmetic restriction is indeed necessary to rule out non-oscillation of an individual pseudoabelian integral $I(t)$, there would be almost no hope for uniform bounds for the number of zeros of pseudoabelian integrals.

1.3. **Principal result.** Our main result is the *uniform non-oscillation* of pseudoabelian integrals associated with a *generic* Darbouxian integrable foliation near the separatrix polygon $S(\theta)$. To formulate it accurately, we introduce Darbouxian classes of integrable systems. Let $m_1, \dots, m_n$ be a tuple of natural numbers, $m_j \in \mathbb{N}$.

Definition 1. The Darbouxian class $\mathscr{D} = \mathscr{D}(m_1, \dots, m_n)$ is the class of real integrable foliations on $\mathbb{R}P^2$ defined by the Pfaffian equations $\theta = 0$, where θ is a *logarithmic form* with constraints on the degrees as follows[3],

$$\theta = \sum_{j=1}^{n} \lambda_j \frac{dp_j}{p_j}, \qquad p_j \in \mathbb{R}[x, y],\ \deg p_j \leqslant m_j,\ \lambda_j > 0. \tag{1.6}$$

The polynomials p_j are assumed to be irreducible and different, hence the polar locus $S(\theta)$ is the union of n algebraic curves $\{p_j = 0\} \subset \mathbb{R}^2$.

Foliations from a fixed Darbouxian class are parameterized by points from an open subspace in the Euclidean space with the coordinates being the exponents $\lambda_1, \dots, \lambda_n > 0$ and the respective polynomials p_j, identified with their coefficients. Using this identification, one can define open neighborhoods of a given closed form $\theta = 0$ in its Darbouxian class $\mathscr{D}$. We will consider only foliations which have continuous families of ovals accumulating to the separatrix $S(\theta)$ or its proper subset (as usual, in the Hausdorff sense).

If a real rational 1-form ω on $\mathbb{R}^2$ is *admissible* for the logarithmic form θ, i.e., if the polar locus of ω belongs to the separatrix $S(\theta)$, then the integral $I(t) = \oint_{L_t}$ over any non-singular real oval L_t, $t \neq 0$, is well defined. We claim that the number of real isolated zeros of such integrals is locally uniformly bounded.

Theorem 2 (uniform non-oscillation of pseudoabelian integrals)**.** *Let the form $\theta_0 = \sum_{j=1}^{n} \lambda_j \frac{dp_j}{p_j}$ defining the Darbouxian integrable foliation $\mathcal{F} = \{\theta_0 = 0\}$ be from the class $\mathscr{D}(m_1, \dots, m_n)$, and assume that the real algebraic curves $\{p_j = 0\}$ are smooth and intersect transversally (in particular, there are no triple intersections). Let ω_0 be a rational one-form admissible for θ_0 with poles of order at most m_0.*

Then the number of isolated zeroes of the pseudoabelian integral near $t = 0$, as a function of (θ, ω), is locally bounded at (θ_0, ω_0) over all admissible pairs (θ, ω).

In other words, for any admissible pair of rational 1-forms (θ_0, ω_0) as in the Theorem, there exist a finite number $N = N(\theta_0, \omega_0)$ and $\varepsilon > 0$, depending on the pair, such that the number of real isolated zeros of any pseudoabelian integral corresponding to (θ, ω) and to a bounded family of

[3]One can easily show (e.g., see [6]) that a closed rational form θ with poles of order $\leqslant 1$ on $\mathbb{C}P^2$ has the form (1.6) with complex constants λ_j.

cycles L_t, $t \in (0, \varepsilon)$, is no greater than N provided that the pair (θ, ω) is sufficiently close to (θ_0, ω_0).

After this result one may expect that the number of zeros of pseudoabelian integrals would be uniformly bounded over all Darbouxian classes. The following Conjecture is a relaxed form of the Infinitesimal Hilbert 16th Problem (Conjecture 1)

Conjecture 2. *For any Darbouxian class $\mathscr{D}(m_1, \dots, m_n)$ and any degree m_0 there exists a finite number $N = N(m_0, m_1, \dots, m_n)$ such that any pseudoabelian integral of a rational 1-form ω of degree $\leqslant m_0$ along ovals of the corresponding Darbouxian integral* (1.3) *has no more than N real isolated zeros.*

Again in contrast with the Hamiltonian case, the Conjecture does not follow from Theorem 2, since the compactness arguments fail for pseudoabelian integrals. Indeed, the assumption on the separatrix is an open condition in $\mathscr{D}$ which in the limit may degenerate into non-transversal singularities at the corners. Besides, the Darbouxian classes themselves are non-compact and in the limit contain closed forms with poles of higher orders and "vanishing separatrices" as $\lambda_j \to 0^+$. All these scenarios require additional study.

1.4. **Proofs: Mellin transform and generalized Petrov operators.** The most fundamental tool used to prove nonoscillation-type results, is the classical derivation–division algorithm [15] which goes as far back as to Descartes. In the modern language it consists in constructing a differential operator $\mathcal{D}$ (in general, with variable coefficients) such that the result of the derivation $\mathcal{D}I(t)$ is a function without real isolated zeros (e.g., identical zero). By the Rolle-Descartes theorem, the number of zeros of zeros of I is then bounded by a constant explicitly expressible in terms of $\mathcal{D}$ (in the simplest case of operators with constant coefficients, by the order of $\mathcal{D}$, provided that all characteristic roots are real). This observation was generalized in several directions, among them for the vector-valued [3] and complex analytic [11] functions.

However, this method encounters serious difficulties when applied to parametric families of functions, especially the families which admit *asymptotic* (rather than convergent) representation as in (1.5) as $t \to 0^+$, with the exponents μ_j depending on parameters and eventually coinciding between themselves. The operator $\mathcal{D}$, mentioned above, turns out to be very sensitive to these parameters.

Our proof is based on introducing a new class of real *pseudo*differential operators $\mathcal{P}$ which decrease in a controlled way the number of isolated zeros. More precisely, for each such operator $\mathcal{P}$ and any family F of functions analytic on the universal cover of the closed unit disk and analytically depending on parameter there exists a finite number ("Rolle index") $N = N(\mathcal{P}, \mathrm{F})$ such that for any $u = u(t) \in \mathrm{F}$ on $(0, 1)$ the numbers of real isolated zeros of u

and $\mathcal{P}u$ are related by the "Rolle inequality"

$$\#\{u = 0\} \leqslant \#\{\mathcal{P}u = 0\} + N. \tag{1.7}$$

These operators, depending on auxiliary parameters, in a sense interpolate between the "usual" differential operators, the Petrov difference operators that were used implicitly in [13] and explicitly in [14], and the operator of "taking imaginary part" which played the key role in [10].

The operator $\mathcal{P}$ is defined via the *Mellin transform* which associates with each function $u(t)$ (say, defined on the interval $[0,1]$), the function $v(s)$ of a complex argument s by the formula

$$v = \mathscr{M}u, \qquad v(s) = \int_0^1 t^{s-1}u(t)\,dt. \tag{1.8}$$

We show (Theorem 6 below) that the Mellin transform $\mathscr{M}I$ of a pseudoabelian integral $I(t)$ is a function holomorphic in the right half-plane $\operatorname{Re} s > s_0$, which admits analytic continuation as a meromorphic function in the entire plane $s \in \mathbb{C}$ with poles located on finitely many real arithmetic progressions on the shifted negative semiaxis $\mathbb{R}_-$. The distance between subsequent poles may be very small, depending on the arithmetic of the exponents λ_j, which results in the divergence of the representation (1.5).

The operator P is constructed as

$$P = \mathscr{M}^{-1}\mathcal{K}\mathscr{M}, \qquad \mathcal{K} : v(s) \mapsto K(s)v(s), \tag{1.9}$$

where $K(s)$ is an entire function with zeros on these progressions. Then the product $K \cdot \mathscr{M}I$ becomes an entire function whose inverse Mellin transform $\mathscr{M}^{-1}$ is identically zero. Note that a differential operator with constant coefficients (the class sufficient for many applications of the derivation–division algorithm) can be represented in this way with a *polynomial* kernel $K(s)$: this explains the connections with the traditional method.

Flexibility of the construction of the operator $\mathcal{P}$ (location of zeros of $K(s)$) allows to choose $\mathcal{P} = \mathcal{P}_\theta$ in a way *analytically depending* on the form $\theta \in \mathscr{D}$ such that $\mathcal{P}_\theta I \equiv 0$ for any pseudoabelian integral $I = I_\theta(t)$ associated with the integrable foliation $\theta = 0$.

The cornerstone of the proof is the demonstration of the Rolle inequality (1.7) for the operator $\mathcal{P}$. We prove that the corresponding constant $N = N(\mathcal{P}_\theta)$ depends in a uniformly bounded way on the logarithmic 1-form $\theta \in \mathscr{D}$. This is sufficient for the proof of Theorem 2.

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2. Analytic representation of pseudoabelian integrals

2.1. **Parametrization of admissible pairs of forms.** We define the complexification $\mathscr{D}_{\mathbb{C}} = \mathscr{D}_{\mathbb{C}}(m_1, ..., m_n)$ of $\mathscr{D}$ as the space of logarithmic forms θ,

$$\theta = \sum_{j=1}^{n} \lambda_j \frac{dp_j}{p_j}, \qquad \lambda_j \in \mathbb{C},\ p_j \in \mathbb{C}[x,y],\ \deg p_j \leqslant m_j,$$

where, as above, p_j are assumed to be coprime and irreducible.

Denote by $\mathscr{D}'$ the subset $\mathscr{D}' \subset \mathscr{D} \subset \mathscr{D}_{\mathbb{C}}$ of all $\theta \in \mathscr{D}$ with smooth real algebraic curves $\{p_j = 0\}$ intersecting transversally.

The pairs (θ, ω) with $\theta \in \mathscr{D}'$ and real one-form ω admissible for θ with poles of order less than m_0 form a set $\mathscr{B} = \mathscr{B}(m_0, m_1, ..., m_n)$, which has an evident topology of a real analytic vector bundle with base $\mathscr{D}'$.

The set $\mathscr{B}$ is a real analytic subvariety of the set $\mathscr{B}_{\mathbb{C}}$, defined as the set of pairs of complex one-forms (θ, ω), with $\theta \in \mathscr{D}_{\mathbb{C}}$ and ω admissible for θ with poles of order less than m_0. The set $\mathscr{B}_{\mathbb{C}}$ has an evident structure of a complex analytic vector bundle over complex analytic manifold $\mathscr{D}_{\mathbb{C}}$.

2.2. **Local linearization of logarithmic foliations.** We start with a well-known fact whose demonstration is only briefly sketched for completeness of exposition.

Lemma 1. *A closed meromorphic 1-form θ with first order poles on two transversally intersecting analytic curves in $\mathbb{C}^2$ can be analytically linearized near the point of their intersection. More exactly, there is a linearizing transformation ϕ mapping a neighborhood of a closed polydisk $\{|x|, |y| \leqslant 2\} \subset \mathbb{C}^2$ to a neighborhood of the point of the intersection, such that the pull-back $\phi^*\theta$ is the logarithmic one-form $\lambda\frac{dx}{x} + \mu\frac{dy}{y}$ defined in this neighborhood.*

This linearizing transformation depends analytically on the complex form $\theta \in \mathscr{D}_{\mathbb{C}}$.

Proof. The two curves can be simultaneously rectified to become the coordinate axes. The local fundamental group of the complement is commutative, and is generated by two loops around these axes: this means that there exist two (complex) numbers λ, μ (the residues of θ) such that the difference $\theta - (\lambda\frac{dx}{x} + \mu\frac{dy}{y})$ is exact, the differential of a meromorphic function. Because of the constraints on the order of the poles, this difference is the differential of a holomorphic function dg, holomorphically depending on θ. The inverse of the holomorphic transformation $(x, y) \mapsto (x, y \exp(\mu^{-1}g))$ brings the form θ to the required "linear" normal form in some neighborhood of the origin, which can be subsequently expanded to the polydisk by a linear dilatation in x, y. $\square$

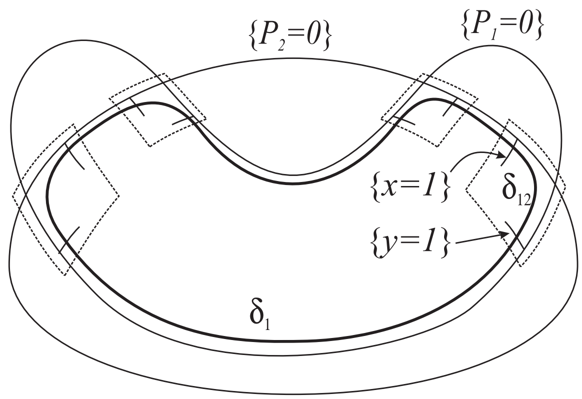

FIGURE 1. Splitting of a cycle into pieces.

Here is the linearization transformation written explicitly in terms of the first integral. Let

$$f = Cp_1(x,y)^{\lambda} p_2(x,y)^{\mu} \prod_{j=3}^{n} p_j(x,y)^{\lambda_j} \tag{2.1}$$

be a first integral of θ, $C \neq 0$, and assume that the transversally intersecting curves are $\{p_1 = 0\}$ and $\{p_2 = 0\}$. The inverse of the mapping

$$(x,y) \to (\tilde{x}, \tilde{y}), \quad \text{where } \tilde{x} = p_1(x,y),\ \tilde{y} = p_2 \prod_{j=3}^{n} p_j(x,y)^{\lambda_j/\mu}$$

is the required diffeomorphism ϕ.

Lemma 2. *For any C bigger than some $C_0 \gg 1$ the dilatation can be chosen in such a way that the first integral* (2.1) *becomes a monomial $\tilde{x}^{\lambda}\tilde{y}^{\mu}$ in the new coordinates.*

Indeed, after linearization as in Lemma 1 the first integral becomes const $\tilde{x}^{\lambda}\tilde{y}^{\mu}$, and if const > 1, then an additional dilatation brings the first integral to the required form. □

2.3. **Representation for pseudoabelian integral.**

2.3.1. *Local computation near saddles.* We compute $I(t)$ by evaluating separately integrals along pieces δ_{ij} of L_t lying near saddles $\{p_i = p_j = 0\}$ and along the pieces δ_j of L_t lying near smooth pieces of $\{p_j = 0\}$.

Integrals of monomial 1-forms along the piece δ_{ij} of the integral curve $L_t = \{f = t\}$ lying between two cross-sections, $\{x = 1\}$ and $\{y = 1\}$, can be explicitly computed. Indeed, in the linearized coordinates (x, y) the curve δ_{ij} is, by Lemma 2, given by the explicit formula

$$\delta_{ij} = \{y = t^{1/\mu} x^{-\lambda/\mu}\}, \qquad t \in (0,1), x \in (t^{1/\lambda}, 1), \tag{2.2}$$

eventually after replacing f by Cf, with $C \gg 1$. Therefore the integration of $\omega = x^{p-1}y^q\,dx$, $(p,q) \in \mathbb{Z}^2$, is simple:

$$\int_{t^{1/\lambda}}^{1} x^{p-1}y^q\,dx = t^{q/\mu}\,\frac{x^{p-q\lambda/\mu}}{p-q\lambda/\mu}\bigg|_{t^{1/\lambda}}^{1} = \lambda^{-1}\frac{t^{q/\mu}-t^{p/\lambda}}{p/\lambda-q/\mu}, \tag{2.3}$$

if $q/\mu \neq p/\lambda$. For the resonant case $q/\mu = p/\lambda$ the same computation yields the answer $-\lambda^{-1}t^{q/\mu}\log t$ which can be otherwise obtained by direct passage to limit. Computation of $\int_{\delta_{ij}} x^p y^{q-1}\,dy$ is similar.

Denote by $\ell_{pq\lambda\mu}(t)$ the following elementary function:

$$\ell_{pq\lambda\mu}(t) = \begin{cases} \dfrac{t^{p/\lambda}-t^{q/\mu}}{p/\lambda-q/\mu}, & \text{if } p/\lambda \neq q/\mu, \\ t^{p/\lambda}\log t & \text{otherwise,} \end{cases} \tag{2.4}$$

One can verify that $\ell_{pq\lambda\mu}(t)$ depends analytically on λ,μ for any $t>0$.

2.3.2. *Analytic representation of pseudoabelian integral.*

Lemma 3. *After some rescaling of t, the pseudoabelian integral $I(t)$ corresponding to (θ,ω) sufficiently close to (θ_0,ω_0) in $\mathscr{B}$ admits the representation near the point $t=0$*

$$I(t) = \sum_{p,q,\lambda,\mu} a_{pq\lambda\mu}\ell_{pq\lambda\mu}(t) + \sum_{r,\lambda} b_{r\lambda}t^{r/\lambda}, \tag{2.5}$$

where p,q,r run over a translated octant $\mathbb{Z}^3_+$, $p,q,r \in -m_0+\mathbb{N}$, and the real indices λ,μ run independently over the set $\Lambda = \{\lambda_1,\dots,\lambda_n\} \subset \mathbb{R}^n_+$ of the Darboux exponents of the closed 1-form θ.

Moreover, the coefficients $a_{pq\lambda\mu}, b_{r\lambda}$ depend analytically on the forms θ,ω, and can be continued holomorphically to some neighborhood U of $(\theta_0,\omega_0) \in \mathscr{B}_{\mathbb{C}}$, and continuously to its closure $\overline{U} \Subset \mathscr{B}_{\mathbb{C}}$. The coefficients $a_{\bullet\lambda\mu}, b_{\bullet\lambda}$ decrease at least exponentially: there exist $C,\rho>2$ such that the following bounds hold in $\overline{U}$

$$|a_{pq\lambda\mu}| \leqslant C\rho^{-(p+q)}, |b_{r\lambda}| \leqslant C\rho^{-r}. \tag{2.6}$$

Proof. By assumptions, the Hausdorff limit $\overline{L}_0 = \lim_{t\to 0^-} L_t \subset S(\theta)$ of the real ovals is a bounded curvilinear separatrix polygon, whose edges and vertices depend analytically on $\theta \in \mathscr{D}$ in a sufficiently small neighborhood of θ_0 by the Implicit Function Theorem.

Near each vertex one can linearize the foliation using Corollary 1 in such a way that the first integral is a monomial in the new coordinates, $f = x^\lambda y^\mu$, eventually after a rescaling of f.

The linearizing transformation brings the rational form ω into a meromorphic form in the bidisk $\{|x|,|y| \leqslant 2\}$ with poles of order less than m_0 on the coordinate axes. Such form can be represented as a series

$$\sum_{p,q>-m_0} \alpha_{pq}x^{p-1}y^q\,dx + \beta_{pq}x^p y^{q-1}\,dy$$

converging in the bidisk $\{|x|,|y| \leqslant 2\}$. This convergence implies the exponentially decreasing upper bounds (2.6) on the Taylor coefficients α_{pq},β_{pq} of the form ω in the linearizing coordinates x,y. Since the linearization depends analytically on θ in a closure $\bar{U} \Subset \mathscr{B}_{\mathbb{C}}$ of a sufficiently small neighborhood U of $\theta_0 \in \mathscr{B}_{\mathbb{C}}$ by Lemma 1, the coefficients $\alpha_{pq},\beta pq$ are analytic in $\overline{U}$ and the upper bounds are uniform in this neighborhood. Integrating the series termwise along $\delta_{ij}(t) = \{f = t\} \cap \{|x|,|y| \leqslant 2\}$ as in (2.3), and summing up over all vertices of $\overline{L}_0$ corresponding to the pair (λ,μ) (their number is finite by Bezout theorem) we obtain the first sum in (2.5) and the exponential upper bounds (2.6) for $a_{pq\lambda\mu}$, as finite linear combinations of α_{pq},β_{pq}.

The local cross-sections near the corners given in linearized coordinates by $\{x = 1\}$ and $\{y = 1\}$ become analytic curves transversal to $S(\theta)$ in the initial coordinates on the plane. The integral of the form ω along arcs of leaves L_t between the two cross-sections near two endpoints of the jth edge (on the curve $\{p_j = 0\}$), depends analytically on the arc, as the latter remains nonsingular. In the chart t obtained by restriction of the Darbouxian integral, this means that the corresponding contribution to the integral is an analytic function of t^{1/λ_j}, where λ_j is the residue (Darboux exponent) corresponding to the edge. Again, after a rescaling one can assume that this analytic function converges in a small neighborhood of $\{|t^{1/\lambda_j}| \leqslant 2\}$, uniformly over a small neighborhood of $(\theta_0,\omega_0) \in \mathscr{B}_{\mathbb{C}}$, thus giving the second inequality in (2.6). □

3. The class $\mathfrak{J}$ and its Mellin transform

We introduce a class $\mathfrak{J}$ of functions of one variable as sums of series of type (2.5) with coefficients satisfying (2.6), and define the notion of analytic $\mathfrak{J}$-family in Section 4 using the statement of Lemma 3 as its definition. We prove in Section 5 that the number of zeros on $[0,1]$ is uniformly bounded for any analytic $\mathfrak{J}$-family.

3.1. **Definition of the class $\mathfrak{J}$.**

Definition 2. We define $\mathfrak{J}$-series as a formal series σ of the form

$$\sum_{p,q,\lambda,\mu} a_{pq\lambda\mu}\ell_{pq\lambda\mu}(t) + \sum_{r,\lambda} b_{r\lambda}t^{r/\lambda}, \qquad a_{pq\lambda\mu}, b_{r\lambda} \in \mathbb{C}, \tag{3.1}$$

where $\lambda,\mu \in \Lambda = \{\lambda_1,...,\lambda_n\} \subset \mathbb{R}_+$ - a finite set, $p,q,r \in -m_\sigma + \mathbb{N}$, and

$$|a_{pq\lambda\mu}| < C\rho^{-p-q}, |b_{r\lambda}| < C\rho^{-r} \quad \text{for some } C,\rho > 2. \tag{3.2}$$

The set Λ is called the spectrum of σ.

We denote further by $M = M(\sigma)$ some number smaller than all exponents in (3.1), e.g.:

$$M = \min_i(-m_\sigma\lambda_i). \tag{3.3}$$

The class $\mathfrak{J}$ is the class of functions $f(t)$ in one variable $t \in [0,1]$, called $\mathfrak{J}$-functions, representable as a sum of some $\mathfrak{J}$-series.

One should emphasize that the $\mathfrak{I}$-series corresponding to a function $f \in \mathfrak{I}$ is not unique: indeed, even the function $\ell_{pq\lambda\mu}(t)$ can be written as a sum of two monomials. Also, we do not know any effective way to check if a given function belongs to $\mathfrak{I}$. However, according to Lemma 3, the pseudoabelian integrals belong to $\mathfrak{I}$.

3.2. **Analytic continuation of $\mathfrak{I}$-functions.** We prove here that $\mathfrak{I}$-functions can be analytically continued to the universal cover of the punctured unit disk.

We start with an elementary inequality.

Lemma 4. *For any $t, \alpha, \beta \in \mathbb{C}$ we have*

$$\left|\frac{t^\alpha - t^\beta}{\alpha - \beta}\right| \leqslant |t^\gamma||\log t|$$

for some $\gamma \in [\alpha, \beta] \subset \mathbb{C}$.

Indeed, take $z = \log t$ in the following inequality

$$\left|e^{\alpha z} - e^{\beta z}\right| = \left|z \int_\beta^\alpha e^{wz} dw\right| \leqslant |\alpha - \beta| \cdot |e^{\gamma z}| \cdot |z| \quad \text{for some } \gamma \in [\beta, \alpha]. \quad \square$$

Let $\widetilde{\mathbb{C}^*}$ denote the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let $\tilde{D}^\circ \subset \widetilde{\mathbb{C}^*}$ be the universal cover of the punctured closed unit disk.

Lemma 5. *For any $A > 0$ the series* (3.1) *converges uniformly after multiplication by t^{-M} in a neighborhood in $\widetilde{\mathbb{C}^*}$ of the sector $S_A = \{|t| \leqslant 1, |\arg t| \leqslant A\} \subset \tilde{D}^\circ$, and, in particular, on* $[0, 1]$.

Lemma 4 implies that for some constant C_A and for all p, q, λ, μ in (3.1) we have $|t^{-M}\ell_{pq\lambda\mu}(t)| \leqslant C_A 2^{(p+q)}$ as long as $|t|^{1/\lambda}, |t|^{1/\mu} < 2$. These inequalities define a neighborhood of S_A in which the series (3.1) converges uniformly by (3.2). $\square$

Corollary 1. *$\mathfrak{I}$-functions are real analytic functions on* $(0, 1]$ *admitting analytic continuation to the universal cover of the punctured closed unit disc.*

Remark 1. If the ratios of all consecutive Darboux exponents λ_i/λ_j, $i, j = 1, \dots, n$ are either rational, or "nice" irrational (not admitting too close approximations by rational numbers so that the differences $p\lambda_i - q\lambda_j$ decrease no faster than exponentially with $p, q \to +\infty$), then the series (2.5) can be re-expanded in powers of t^{1/λ_j}, and this re-expansion *converges*. Then the integral $I(t)$ becomes representable under the form (1.5). Application of the standard fewnomial technique allowed C. Moura to prove that for such collections of the Darboux exponents the pseudoabelian integral $I(t)$ is non-oscillating, moreover, that the number of isolated zeros is locally uniformly bounded over ω, *provided that the Darbouxian exponents are fixed* [8]. However, variation of the exponents clearly destroys the arithmetic conditions.

Remark 2. The transcendental binomials $\ell_{pq\lambda\mu}$ which in the limit converge to logarithms, first appeared at least as early as in 1951 in the work by E. Leontovich [7] on bifurcations of separatrix loop. Since then they regularly re-appear in different disguises, most recently as the so called Ecalle–Roussarie compensator.

This ubiquity can be probably explained by the fact that the function $\ell_\varepsilon(t) = t(t^\varepsilon - 1)/\varepsilon$ is in a sense the only possible Pfaffian deformation of the "logarithm" $\ell_0(t) = -t\log t$. Indeed, the graph of the latter function is an integral curve of the linear equation $\xi_0 = 0$, where $\xi_0 = t\,dy + (y-t)\,dt$ is a Pfaffian form. This linear Pfaffian equation has a singularity of the Poincaré type (resonant node) at the origin. Any (nonlinear) deformation ξ_ε of the form ξ_0 is necessarily analytically linearizable by the Poincaré theorem, the linearizing chart depending analytically on the parameter ε. Among integral curves of the perturbed linear system one is the graph of ℓ_ε.

3.3. **Mellin transform of $\mathfrak{J}$.** The (one-sided) Mellin transform of a function $u(t) \in L^1_{loc}((0,1])$ defined on the interval $(0,1]$ is the function $v = \mathscr{M}u$ of complex variable s defined by the integral transform (1.8):

$$v(s) = \mathscr{M}u(s) = \int_0^1 t^{s-1}u(t)dt.$$

If the function u grows moderately as $t \to 0^+$, $|u(t)| = o(t^{-M})$ for some finite M, then the integral in (1.8) converges uniformly on compact subsets of the half-plane $\{\operatorname{Re} s > M\}$. In particular, Mellin transforms are defined for all $\mathfrak{J}$-functions.

The formal Mellin transform of a $\mathfrak{J}$-series is defined by its action on the basic functions

$$\mathscr{M}(t^{r/\lambda}) = \frac{1}{s + r/\lambda}, \quad \mathscr{M}(\ell_{pq\lambda\mu}) = \frac{1}{(s+p/\lambda)(s+q/\mu)}$$

and extended by linearity, so the result is a formal sum of these rational functions. We prove that this formal sum converges to a meromorphic function which is an analytic continuation of the Mellin transform of the sum of the $\mathfrak{J}$-series.

Lemma 6. *The Mellin transform $g = \mathscr{M}f$ of a function $f \in \mathfrak{J}$ given as a sum of the series* (3.1) *is the sum of the series:*

$$g(s) = \sum_{p,q,\lambda,\mu} \frac{a_{pq\lambda\mu}}{(s+p/\lambda)(s+q/\mu)} + \sum_{r,\lambda} \frac{b_{r\lambda}}{s + r/\lambda}, \tag{3.4}$$

with the same p, q, r, λ, μ and $a_{pq\lambda\mu}, b_{r\lambda}$ as in Definition 2.

In particular, $g(s)$ extends analytically as a meromorphic function on the entire complex plane $\mathbb{C}$, the poles of $g(s)$ are of at most second order, are all real and lie in the union of n arithmetic progressions of the form $\cup_{i=1}^n \lambda_i^{-1}(m_0 - \mathbb{N})$.

Proof. The Mellin transforms of each elementary binomial $\ell_{pq\lambda\mu}$ and of $t^{r/\lambda}$ are equal to its formal Mellin transform. Termwise integration of right-hand side of (3.1) multiplied by t^{s-1} is possible due to the uniform convergence on

$[0,1]$ of this product for $\operatorname{Re} s > -M$ by Corollary 5. Therefore we get (3.4) in $\{\operatorname{Re} s > -M\}$. This double series converges to a meromorphic function on $\mathbb{C}$ given by the same formula. □

Let γ_M be the boundary of the semistrip $\{z | \operatorname{Re} z \leqslant -M+1, |\operatorname{Im} z| \leqslant -1\}$ oriented counterclockwise, with M defined in (3.3).

For any function bounded on γ_M we define an integral transform $\mathcal{C}_M$ as:

$$\mathcal{C}_M g(t) = \frac{1}{2\pi i} \int_\gamma t^{-s} g(s) ds. \tag{3.5}$$

For a meromorphic in $\mathbb{C}$ function g bounded on $\{|\operatorname{Im} s| = 1\}$ with poles on the real line and bounded from the right we define $\mathcal{C}g$ as $\lim_{M\to-\infty} \mathcal{C}_M g$.

Theorem 3. *The Mellin transform $\mathscr{M} f$ of any function $f \in \mathfrak{J}$ is bounded on γ_M, where $M = M(\sigma)$ for some $\mathfrak{J}$-series representing f.*

The restriction of the integral transform $\mathcal{C}$ to $\mathscr{M}\mathfrak{J}$ is the inverse operator to the Mellin transform $\mathscr{M}$.

For $\varkappa \in \mathbb{R}$ and $f \in \mathfrak{J}$ the transform $\mathcal{C}$ of $e^{-i\varkappa s}\mathscr{M} f(s)$ is defined and is equal to $f(e^{i\varkappa}t)$:

$$\mathcal{C}(e^{-i\varkappa s}\mathscr{M} f(s)) = f(e^{i\varkappa}t). \tag{3.6}$$

Abusing the language we call $\mathcal{C}$ the inverse Mellin transform, though its domain of definition is bigger than $\mathscr{M}\mathfrak{J}$.

Proof. Evidently $|s + p/\lambda| \geqslant 1$ on γ, so the convergence of (3.4) is uniform and $|g| \leqslant \sum_{p,q,\lambda,\mu} |a_{pq\lambda\mu}| + \sum_{r,\lambda} |b_{r\lambda}| < \infty$ on γ.

The kernel t^{-s} decreases exponentially on γ for $|t| < 1$, so we can integrate $t^{-s}g(s)$ termwise on γ. Calculating the residues, we see that

$$\mathcal{C}\left(\frac{1}{s + r/\lambda}\right) = t^{r/\lambda}, \quad \mathcal{C}\left(\frac{1}{(s + p/\lambda)(s + q/\mu)}\right) = \ell_{pq\lambda\mu},$$

so the Mellin transform and the inverse Mellin transform translate (3.1) to (3.4) and vice versa, i.e. are mutually inverse if restricted to $\mathfrak{J}$ and $\mathscr{M}\mathfrak{J}$ correspondingly.

Since $e^{-i\varkappa s}$ is bounded by $e^{|\varkappa|}$ on γ, one can compute the inverse Mellin transform (3.5) termwise, again due to the uniform convergence of the series for $g(s)$ and the exponential decay of t^{-s}. For $\ell_{pq\lambda\mu}$ and t^α the (3.6) can be checked immediately. □

3.4. **Asymptotic series for $\mathfrak{J}$-functions and quasianalyticity of $\mathfrak{J}$.** Let $f \in \mathfrak{J}$ be a sum of some $\mathfrak{J}$-series σ. Any $\mathfrak{J}$-series σ can be expanded into a formal series by powers of t:

$$\check{f} = \sum_\alpha t^\alpha \left(c_\alpha^{-1} + c_\alpha^{-2} \log t\right), \qquad \alpha \in (-m_\sigma + \mathbb{N})\Lambda^{-1}, \tag{3.7}$$

where Λ is the spectrum of σ and Λ^{-1} denotes the set of its reciprocals. We prove below that the result of the expansion does not depend on the choice

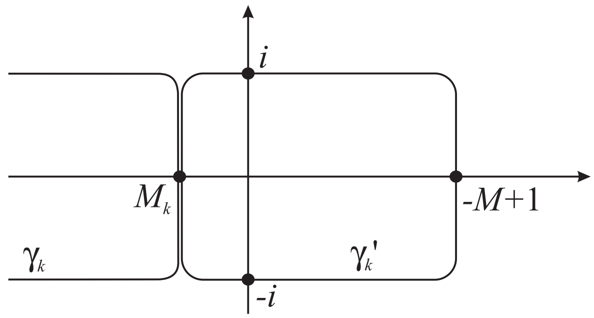

FIGURE 2. Splitting of the integration contour γ into two.

of σ and represents the asymptotic series for f. Moreover, the class $\mathfrak{J}$ turns out to be quasianalytic at 0: any two functions having the same asymptotic series coincide.

Lemma 7. *The coefficients $c_\alpha^{-1}, c_\alpha^{-2}$ are the Laurent coefficients of the Mellin transform $g = \mathscr{M} f$ of f at the pole $s = \alpha$.*

The series $\check{f}$ is an asymptotic series for f, so does not depend on σ. If $\check{f} \equiv 0$ then $f \equiv 0$. Equivalently, the only entire function in $\mathscr{M}\mathfrak{J}$ is zero.

Proof. The first claim follows because the Laurent coefficients of g are given by exactly the same formulae as c_α, d_α:

$$c_\alpha^{-1} = \sum_{\substack{q,\mu \\ q/\mu=\alpha}} b_{q\mu} + \sum_{\substack{p,q,\lambda,\mu \\ q/\mu\neq p/\lambda=\alpha}} \frac{a_{pq\lambda\mu}}{p/\lambda - q/\mu}, \quad c_\alpha^{-2} = \sum_{\substack{p,q,\lambda\mu \\ q/\mu=p/\lambda=\alpha}} a_{pq\lambda\mu},$$

both converging due to (3.2).

Choose some $0 < \lambda_0 \leqslant \min \Lambda$. Each segment $[k\lambda_0, (k+1)\lambda_0)$, $k \in \mathbb{Z}$, has at most n points common with $\Lambda\mathbb{Z}$, so one can choose M_k from this segment at least $\lambda_0/2n$-distant from $\Lambda\mathbb{Z}$.

This implies that $|z + p/\lambda| \geqslant \lambda_0/2n$ on $\operatorname{Re} z = M_k$, so $|g| \leqslant C$ on the boundary γ_k of the semistrip $\{\operatorname{Re} z \leqslant M_k, |\operatorname{Im} z| \leqslant 1\}$, and the main point is that C is independent of k. Denote by γ_k' the boundary of the rectangular $\{M_k \leqslant \operatorname{Re} z \leqslant -M+1, |\operatorname{Im} z| \leqslant 1\}$ and let split integration along γ in the definition of $\mathcal{C}$ into a sum of two integrals, one along γ_k and another along γ_k', see Figure 3.4.

Evaluation of the integral along γ_k' gives a finite partial sum of $\check{f}$:

$$\frac{1}{2\pi i} \int_{\gamma_k'} t^{-s} g(s) ds = \sum_{\alpha < -M_k} t^\alpha \left(c_\alpha + d_\alpha \log t\right).$$

Integral along γ_k can be estimated from above by $2Ct^{-M_k}(1 + 1/\log t)$, which proves the second claim.

If $\check{f} \equiv 0$, then g has no poles at all, i.e. g is entire. Therefore the integral along γ_k' is zero for all k. Therefore $|f(t)| \leqslant 2Ct^{-M_k}(1 + 1/\log t)$ for all $t \in (0,1)$, and for all M_k. Since $M_k \to -\infty$ as $k \to -\infty$ and C

is independent of k, we conclude that $f(t) \equiv 0$ on $(0,1)$, and therefore everywhere by analyticity. This means that $g = \mathscr{M} f \equiv 0$ as well. $\square$

4. Analytic $\mathscr{M}\mathfrak{J}$-families and operator $\mathcal{E}_\varkappa$

Definition 3. Let $V \Subset \mathbb{R}^n$ be some compact set, and let $U \subset \mathbb{C}^n$ be a bounded neighborhood of V. We denote by $\mathfrak{O}_{V\subset U}$ the space of real analytic functions on V which can be extended holomorphically to U and continuously to the closure $\overline{U}$ of U, equipped with the norm $\|u\|_{\mathfrak{O}} := \max_{\nu\in\overline{U}} |u(\nu|)$.

Lemma 8. *The space $\mathfrak{O}_{V\subset U}$ is complete.* $\square$

Definition 4. Consider a family $\check{\mathrm{F}} = \{\sigma_\nu\}$ of $\mathfrak{J}$-series, $\nu \in V \Subset \mathbb{R}^N$, and denote by $\Lambda(\nu) = \{\lambda_1(\nu), ..., \lambda_n(\nu)\} \subset \mathbb{R}_+$ the spectrum of the $\mathfrak{J}$-series $\sigma(\nu)$.

We say that $\check{\mathrm{F}}$ is an analytic family of $\mathfrak{J}$-series on pair $V \subset U$ for some bounded neighborhood $U \subset \mathbb{C}^N$ of V in $\mathbb{C}^N$ if the functions $\lambda_i(\nu)$, $a_{pq\lambda(\nu)\mu(\nu)}(\nu)$, $b_{r\lambda(\nu)}(\nu)$ belong to $\mathfrak{O}_{V\subset U}$, the analytic continuation of $\lambda_i(\nu)$ does not vanish in $\overline{U}$ (i.e. $\lambda_i^{-1}(\nu) \in \mathfrak{O}_{V\subset U}$ as well), and

$$\left\|a_{pq\lambda(\nu)\mu(\nu)}(\nu)\right\|_{\mathfrak{O}}| \leqslant C\rho^{-(p+q)}, \quad \left\|b_{r\lambda(\nu)}(\nu)\right\|_{\mathfrak{O}} \leqslant C\rho^{-r} \tag{4.1}$$

for some $C = C(\check{\mathrm{F}}), \rho = \rho(\check{\mathrm{F}}) > 2$.

An analytic $\mathfrak{J}$-family $\mathrm{F} = \{f(t;\nu)\} \subset \mathfrak{J}$ is a family of sums of an analytic family of $\mathfrak{J}$-series. In other words, for every $f(t;\nu)$ one can choose a $\mathfrak{J}$-series σ_ν such that $f(t;\nu)$ is a sum of σ_ν, and $\{\sigma_\nu\}$ is an analytic family of $\mathfrak{J}$-series.

Analytic $\mathscr{M}\mathfrak{J}$-family is by definition the Mellin transform of an analytic $\mathfrak{J}$-family.

Lemma 9. *For a given analytic family $\check{\mathrm{F}}$ of $\mathfrak{J}$-series the number m_σ is uniformly bounded from above by some number $m_{\mathrm{F}} < \infty$ for all $\sigma = \sigma_\nu \in \check{\mathrm{F}}$.*

Indeed, the sets $S_k = \{\nu \in U | m_{\sigma_\nu} \leqslant k \in \mathbb{N}\}$ are analytic subsets of U and their union is U, so $S_k = U$ for some k. $\square$

Lemma 3 says that the family of pseudoabelian integrals parameterized by $\mathscr{B}$ is locally an analytic $\mathfrak{J}$-family.

Let $\check{\mathrm{F}} = \{\sigma_\nu\}$ be an analytic family of $\mathfrak{J}$-series on pair $V \subset U$, and consider the series

$$\sum_{p,q,\lambda(\nu),\mu(\nu)} a_{pq\lambda(\nu)\mu(\nu)}\ell_{pq\lambda(\nu)\mu(\nu)}(t) + \sum_{r,\lambda(\nu)} b_{r\lambda(\nu)}t^{r/\lambda(\nu)} \quad \text{for} \nu \in U. \tag{4.2}$$

Lemma 10. *For every $A > 0$ there exists some $\varepsilon > 0$ such that this series converges uniformly on compact subsets of*

$$U_\varepsilon = \{|\arg t| \leqslant A, 0 < |t| < e^\varepsilon\} \times \left(\{|\operatorname{Im}\lambda_i^{-1}| < \varepsilon\} \cap U\right) \subset \widetilde{\mathbb{C}^*} \times U,$$

so its sum is a function holomorphic on U_ε.

Proof. In U_ε we get $|t^{\lambda^{-1}(\nu)}| \leqslant \exp\left(\varepsilon\left[A + \|\lambda^{-1}(\nu)\|_{\mathfrak{O}}\right]\right)$, i.e. $|\ell_{pq\lambda\mu}(t)|$ can grow exponentially fast. However, the fast decay of coefficients $a_{pq\lambda\mu}, b_{r\lambda}$ guarantees convergence if ε is sufficiently small: the series (3.1) converges

uniformly on compact subsets of U_ε as soon as $\varepsilon\left[A+\|\lambda_i^{-1}(\nu)\|_{\mathfrak{O}}\right]<\ln 2$ for $i=1,...,n$. $\square$

Remark 3. For functions from $\mathscr{M}\mathfrak{J}$ one can try to define analytic dependence on parameter as analytic dependence of $f(t;\nu)$ on ν on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Analytic $\mathscr{M}\mathfrak{J}$-families depend analytically on parameter in this sense, as shown above, but the opposite implication seems to be wrong.

4.1. **The operator $\mathcal{E}_\varkappa$.** From Lemma 11 below follows that the rotation of argument preserves $\mathfrak{J}$: for any $f(t)\in\mathfrak{J}$ the function $f(e^{i\varkappa}t)$, $\varkappa\in\mathbb{R}$, is also in $\mathfrak{J}$. From (3.6) it seems that the Mellin conjugate to the rotation is the operator of multiplication by $e^{-i\varkappa s}$. However, the latter does not preserves $\mathscr{M}\mathfrak{J}$: the function $e^{-i\varkappa s}\mathscr{M}f(s)$ grows as $|\operatorname{Im}s|\to\infty$, while any function in $\mathscr{M}\mathfrak{J}$ decreases as $|\operatorname{Im}s|^{-1}$.

In this section we define the Mellin conjugate $\mathcal{E}_\varkappa$ of the rotation on $\mathscr{M}\mathfrak{J}$, and prove that application of operator $\mathcal{E}_\varkappa$ with $\varkappa$ analytically depending on parameters preserves $\mathscr{M}\mathfrak{J}$-families.

Let start from the standard computation applied to $f\in\mathfrak{J}$:

$$\begin{aligned}\mathscr{M}(f(e^{i\varkappa}t)) &= \int_0^1 t^{s-1}f(e^{i\varkappa}t)dt\\ &= \int_0^{e^{-i\varkappa}} t^{s-1}f(e^{i\varkappa}t)dt+\int_{C_\varkappa} t^{s-1}f(e^{i\varkappa}t)dt\\ &= e^{-i\varkappa s}\int_0^1 u^{s-1}f(u)du+R(s)=e^{-i\varkappa s}\mathscr{M}f(s)+R(s),\end{aligned}\tag{4.3}$$

where $C_\varkappa=\{e^{iz},z\in[-\varkappa,0]\}$ and $R(s)=\int_{C_\varkappa}t^{s-1}f(t)dt$ is an entire function of exponential type. One can immediately see that $R(s)$ is bounded in $\{|\operatorname{Im}s|\leqslant 1\}$. Therefore $\mathcal{C}R\equiv 0$, which explains (3.6). We take this as a definition of $\mathcal{E}_\varkappa$. In a sense $\mathcal{E}_\varkappa$ is an operator of projection of $e^{-i\varkappa s}\mathscr{M}\mathfrak{J}$ to $\mathscr{M}\mathfrak{J}$ along the space of entire functions.

Definition 5. Let $\varkappa\in\mathbb{R}$. For any $g\in\mathscr{M}\mathfrak{J}$ define $\mathcal{E}_\varkappa g$ as the only function in $\mathscr{M}\mathfrak{J}$ such that the difference $e^{-i\varkappa s}g-\mathcal{E}_\varkappa(g)$ is an entire function.

Remark 4. One can show that $\mathcal{E}_\varkappa g(s)$ is an analytic continuation from $\{\operatorname{Im}s>1\}$ of the integral

$$\frac{1}{2\pi i}\int_\gamma\frac{e^{-i\varkappa z}g}{z-s}dz,$$

where γ is as in (3.5).

Lemma 11. *Let* $\mathrm{F}=\{g(s;\nu)\}$ *be an analytic* $\mathscr{M}\mathfrak{J}$*-family on pair* $V\subset U$*, and let* $\varkappa=\varkappa(\nu)\in\mathfrak{O}_{V\subset U}$*. Then*

(1) $\{\mathcal{E}_{\varkappa(\nu)}g(s;\nu)\}$ *is defined and is an analytic* $\mathscr{M}\mathfrak{J}$*-family on pair* $V\subset\widetilde{U}$ *for some* $\widetilde{U}\subset U$*.*

(2) *the operator* $\mathcal{C}\circ\mathcal{E}_\varkappa\circ\mathscr{M}$ *maps analytic* $\mathfrak{J}$*-families to analytic* $\mathfrak{J}$*-families,*
(3) $\mathcal{C}\mathcal{E}_\varkappa\mathscr{M}f=(e^{i\varkappa}t)$, *and*
(4) *the operator of rotation of argument preserves* $\mathfrak{J}$.

Proof. The uniqueness in Definition 5 follows from Lemma 7: let $g_1,g_2\in\mathscr{M}\mathfrak{J}$ be two functions which differ from $e^{-i\varkappa s}g$ by entire functions. Then their difference is entire and lies in $\mathscr{M}\mathfrak{J}$, which, by Lemma 7, implies that the difference is zero.

Let us construct $\mathcal{E}_\varkappa$. This is simple for $\frac{1}{(s+p/\lambda)(s+q/\mu)}$ and $\frac{1}{s+r/\lambda}$:

$$\frac{e^{-i\varkappa s}}{s-x}=\frac{e^{-i\varkappa x}}{s-x}+\frac{e^{-i\varkappa s}-e^{-i\varkappa x}}{s-x},$$

and

$$2\frac{e^{-i\varkappa s}}{(s-x)(s-y)}=\frac{e^{-i\varkappa x}+e^{-i\varkappa y}}{(s-x)(s-y)}+\frac{e^{-i\varkappa x}-e^{-i\varkappa y}}{x-y}\left(\frac{1}{s-x}+\frac{1}{s-y}\right)+R,$$

where $R=2\frac{(s-x)e^{-i\varkappa y}+(x-y)e^{-i\varkappa s}+(y-s)e^{-i\varkappa x}}{(s-x)(s-y)(x-y)}$ is entire in s.

We extend this by linearity to any $g\in\mathscr{M}\mathfrak{J}$. Namely, for g given by (3.4) we define $\mathcal{E}_\varkappa g(s)$

$$\mathcal{E}_\varkappa g(s)=\sum_{p,q,\lambda,\mu}\frac{\tilde a_{pq\lambda\mu}}{(s+p/\lambda)(s+q/\mu)}+\sum_{r,\lambda}\frac{\tilde b_{r\lambda}}{s+r/\lambda},$$

by the following formulae:

$$\tilde a_{pq\lambda\mu}=\frac12\left(e^{-i\varkappa p/\lambda}+e^{-i\varkappa q/\mu}\right)a_{pq\lambda\mu},\tag{4.4}$$

$$\tilde b_{r\lambda}=e^{-i\varkappa r/\lambda}b_{r\lambda}+\sum_{q,\mu}\frac{e^{-i\varkappa r/\lambda}-e^{-i\varkappa q/\mu}}{r/\lambda-q/\mu}a_{rq\lambda\mu}.\tag{4.5}$$

Take ε so small that $\tilde\rho=\rho e^{-\varepsilon}>2$. Let a neighborhood $\widetilde U\subset U$ of V be so small that $\left|\operatorname{Im}\left(\varkappa(\nu)\lambda_i^{-1}(\nu)\right)\right|<\varepsilon$ for $i=1,...,n$ and $\nu\in\widetilde U$. Then $|e^{-i\varkappa r/\lambda}|<e^{\varepsilon r}$ in $\widetilde U$.

For $p,q,r>-m$ as in (3.4) we have $\max(p,q)<p+q+m$, so for $\nu\in\widetilde U$ we have estimates

$$\frac12\left|e^{-i\varkappa p/\lambda}+e^{-i\varkappa q/\mu}\right|\leqslant e^{\varepsilon(p+q+m)},\quad\left|\frac{e^{-i\varkappa r/\lambda}-e^{-i\varkappa q/\mu}}{r/\lambda-q/\mu}\right|\leqslant e^{\varepsilon(r+q+m)}\|\varkappa\|_{\mathfrak{O}},$$

by Lemma 4.

The upper bounds (3.2) then imply that (4.4),(4.5) converge in $\mathfrak{O}_{V\subset\widetilde U}$, and

$$\tilde a_{pq\lambda\mu}\leqslant\tilde C\tilde\rho^{-p-q},\quad\tilde b_{r\lambda}\leqslant\tilde C\tilde\rho^{-r}\quad\text{in }\widetilde U,$$

which proves the first claim of the Lemma. The second claim is a consequence of the first and of definition of $\mathscr{M}\mathfrak{J}$-family. The third claim follows

from (4.3) and (3.6), so the fourth follows from the third and the fact that $\mathcal{E}_\varkappa$ preserves $\mathscr{M}\mathfrak{J}$. □

5. Petrov operator and proof of Theorem 2

We generalize the so-called *Petrov operator*, see [14], as follows:

$$\mathcal{P}_\varkappa f(t) := \frac{1}{2i}\left(f(te^{-i\varkappa}) - f(te^{i\varkappa})\right) \quad \varkappa \in \mathbb{R} \tag{5.1}$$

Lemma 12. *The inverse Mellin transform of $g(s)\sin(\varkappa s)$ is $\mathcal{P}_\varkappa f$. For any $f \in \mathfrak{J}$*

$$\mathcal{P}_\varkappa f = \mathscr{M}^{-1} \circ (\mathcal{E}_\varkappa - \mathcal{E}_{-\varkappa}) \circ \mathscr{M} f$$

This is a straightforward applications of (3.6) and of Lemma 11 correspondingly. □

The key idea of the proof is a generalization of the Petrov trick invented in [13], where it was used for $\varkappa = \pi$.

Let $f(t)$ be a $\mathfrak{J}$-function. Denote by $N(f)$ the number of isolated zeroes of function f on $(0,1)$ counted with multiplicities, by $\Delta^1_\varkappa(f)$ the increment of $\arg f(t)$ along the arc $\{e^{i\varkappa\phi}, \phi \in [-1,1]\}$ and let $\Delta^0_\varkappa(f)$ be equal to the limit as $\varepsilon \to 0$ of the increment of $\arg f(t)$ along the arc $\{\varepsilon e^{i\varkappa\phi}, \phi \in [-1,1]\}$. Finiteness of all these numbers follows from Lemma 7.

Lemma 13. *For $f \in \mathfrak{J}$ and real on $(0,1)$ the following inequality holds:*

$$N(f) \leqslant 1 + N(\mathcal{P}_\varkappa f) + \frac{1}{2\pi}(\Delta^1_\varkappa(f) + \Delta^0_\varkappa(f)). \tag{5.2}$$

Proof. Applying the argument principle to the sector $\{|\arg z| \leqslant \varkappa, \varepsilon \leqslant |z| \leqslant 1\}$ on the universal cover of the closed unit disc we get

$$2\pi N(f) \leqslant \Delta_- + \Delta_+ + \Delta^1_\varkappa(f) + \Delta^0_\varkappa(f), \tag{5.3}$$

where $\Delta_\pm$ are the increments of the argument of $f(t)$ along the rays $I_\pm = \{te^{\pm i\varkappa}, t \in [\varepsilon, 1]\}$. Since $f(t)$ is real on the real axis, it takes conjugate values on $I_\pm$, so $\Delta_+ = \Delta_-$ and $\operatorname{Im} f|_{I_\pm} = \pm\mathcal{P}_\varkappa f$.

First, assume that $\mathcal{P}_\varkappa f \not\equiv 0$. Evidently, any segment of I_+ where increment of argument of $f(t)$ is greater than π contains at least one zero of $\operatorname{Im} f(t)$, so

$$\Delta_+ \leqslant \pi\left(N(\operatorname{Im} f|_{I_+} + 1\right) = \pi N(\mathcal{P}_\varkappa f) + \pi,$$

and we get the (5.2).

If $\mathcal{P}_\varkappa f \equiv 0$, then, by symmetry, $f(te^{i\varkappa}) = f(te^{-i\varkappa})$, i.e. $f = \tilde{f}(t^{\pi/\varkappa})$ with $\tilde{f}$ holomorphic in a punctured unit disc. Therefore (5.2) follows from the argument principle applied to $\tilde{f}$ and a punctured unit disc. □

Remark 5. As $\varkappa \to 0$, $P_\varkappa f(t)$ tends to a multiple of $f'(t)$, the contributions $\Delta^1_\varkappa(f)$ and $\Delta^0_\varkappa(f)$ of arcs tend to zero, and we get the standard Rolle theorem as a limit case of Lemma 13.

Let $\mathrm{F} = \{f(t,\nu)\}$ be a $\mathfrak{J}$-family on pair $V \subset U$, and let $\varkappa = \varkappa(\nu) \in \mathfrak{O}_{V\subset U}$.

Lemma 14. *The variation $\Delta^1_{\varkappa(\nu)}f$ is uniformly bounded from above by some $\Delta^1=\Delta^1(\mathrm{F},\varkappa)<\infty$ for all $\nu\in V$.*

Proof. The functions $\operatorname{Im} f(e^{i\varkappa(\nu)\phi};\nu)$ and $\operatorname{Re} f(e^{i\varkappa(\nu)\phi};\nu)$ are analytic functions of $\phi\in[-1,1]$ analytically depending on parameter $\nu\in V\Subset\mathbb{R}^n$. Therefore their number of isolated zeros counted with multiplicity is uniformly bounded from above by some number by Gabrielov's theorem [2]. The variation Δ^1 is at most $\pi(\text{this number}+1)$. □

Lemma 15. *The variation $\Delta^0_{\varkappa(\nu)}f$ is uniformly bounded from above by some $\Delta^0=\Delta^0(\mathrm{F},\varkappa)<\infty$ for all $\nu\in V$.*

Proof. For a sum $f\in\mathfrak{J}$ of a $\mathfrak{J}$-series σ the number $\Delta^0_{\varkappa}(f)$ is bounded from above by $-m_\sigma\varkappa$ by Lemma 7. Therefore the existence of the uniform upper bound follows from the Lemma 9. □

Corollary 2. *For F, $\varkappa$ as above and for any $f\in\mathrm{F}$*

$$N(f)\leqslant N(P_\varkappa f)+\Delta(\mathrm{F},\varkappa),\quad \Delta(\mathrm{F},\varkappa)=\Delta^0(\mathrm{F},\varkappa)+\Delta^1(\mathrm{F},\varkappa)+1<\infty. \tag{5.4}$$

5.1. **Proof of Theorem 2.** Starting from the analytic $\mathfrak{J}$-family F of pseudoabelian integrals parameterized by $(\theta,\omega)\in U\subset\mathscr{B}$ we construct the new family $\{P_{\pi\lambda_n(\nu)}f(\bullet,\nu)\}$. It is again an analytic $\mathfrak{J}$-family by Lemma 11. By Corollary 2, application of the operator $P_{\pi\lambda_n(\nu)}$ reduces the number of isolated zeros at most by a finite number uniformly over parameters.

On the other hand, the new family is simpler: its Mellin transform now has poles on $\{m/\lambda_k(\nu),k=1,...,n-1,m\in\mathbb{Z}_{>-m_f}\}$, i.e. on the union of only $n-1$ arithmetic progressions. Indeed, Mellin transform of $\{P_{\pi\lambda_n(\nu)}f_\nu\}$ has the same poles as $\sin(\pi\lambda_n(\nu)s)\mathscr{M}f$, and the first factor has simple zeros at $\{m/\lambda_n(\nu),m\in\mathbb{Z}_{>-m_f}\}$.

Repeating this step n times we arrive at the analytic $\mathfrak{J}$-family whose Mellin transform consists of entire functions. This means that this analytic $\mathfrak{J}$-family is identical zero by Lemma 7, so has no isolated zeros. □.

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